Coordinate Transformations and Triple Integrals

Recall: If we are given a smooth coordinate transformation

$$\begin{cases} x_1 = x_1(u_1, u_2, \ldots, u_n) \\ x_2 = x_2(u_1, u_2, \ldots, u_n) \\ \vdots \\ x_n = x_n(u_1, u_2, \ldots, u_n) \end{cases},$$

the Jacobian of this transform is the determinant of the matrix of first-order partial derivatives of the transform.

Formula $dA_{xy} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$ was a necessary (and often useful) part of our computation of double integrals via coordinate change.

For triple integrals, all of this works exactly the same way. We have $dV_{xyz} = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV_{uvw}$ for transformation

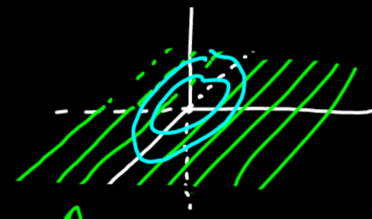
$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w).$$

In what follows, we focus on two specific coordinate systems for $\mathbb{R}^3$ and their corresponding transformations. These are cylindrical coordinates (cylinders ~ boxes) and spherical coordinates (spheres ~ boxes).

Both can be viewed as generalizations of polar coordinates, and can be useful for integrating over "curvy" regions which are not-so-nice in cartesian coordinates...

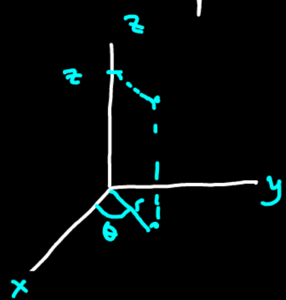
CYLINDRICAL COORDINATES

IDEA: This is a coordinate transformation parametrizing $\mathbb{R}^3$ by using polar coordinates in a plane...

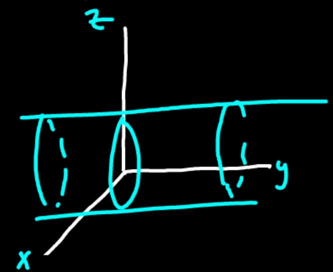


Ex: Here are two examples of cylindrical coordinate systems:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$



$$\begin{cases} x = r\cos(\theta) \\ y = y \\ z = r\sin(\theta) \end{cases}$$



Cylindrical coordinates can be useful for integration.

As such, it will be useful to compute the Jacobian of the transformation $(x,y,z) = (r\cos(\theta), r\sin(\theta), z)$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \det\begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \cos(\theta)\begin{vmatrix} r\cos(\theta) & 0 \\ 0 & 1 \end{vmatrix} - -r\sin(\theta)\begin{vmatrix} \sin(\theta) & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} \sin(\theta) & r\cos(\theta) \\ 0 & 0 \end{vmatrix}$$

$$= r\cos^2(\theta) + r\sin^2(\theta)$$

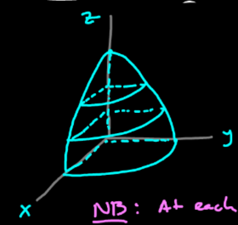
$$= r\left(\cos^2(\theta) + \sin^2(\theta)\right)$$

$$= r$$

NB: $\frac{\partial(x,y,z)}{\partial(r,\theta,w)} = \pm r$ for the cylindrical transformations we'll use...

Ex: Compute $\iiint_R (x+y+z)\,dV$ where R is the solid in the first octant beneath the paraboloid $x^2+y^2+z=4$.

PICTURE



NB: At each fixed z: we have a quarter-circle $r^2 = x^2+y^2 = 4-z$

Sol: Use cylindrical coordinates $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$ $dV = r\,dV_{r\theta z}$

$$R = \left\{ (r,\theta,z) : 0 \le z \le 4,\ 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le \sqrt{4-z} \right\}$$

$$\therefore \iiint_R (x+y+z)\,dV = \int_{z=0}^{4} \int_{r=0}^{\sqrt{4-z}} \int_{\theta=0}^{\frac{\pi}{2}} r\big(r(\cos(\theta)+\sin(\theta)) + z\big)\,d\theta\,dr\,dz$$

Step 1: $$\int_{\theta=0}^{\frac{\pi}{2}} r\big(r(\cos(\theta)+\sin(\theta)) + z\big)\,d\theta = r\Big[r(\sin(\theta)-\cos(\theta)) + z\theta\Big]_{\theta=0}^{\frac{\pi}{2}}$$

$$= r\Big(\big(r(1-0) + \tfrac{\pi}{2}z\big) - \big(r(0-1)+0\big)\Big) = 2r^2 + \tfrac{\pi}{2}rz$$

Step 2: $$\int_{r=0}^{\sqrt{4-z}} \left(2r^2 + \tfrac{\pi}{2}rz\right)dr = \left[\tfrac{2}{3}r^3 + \tfrac{\pi}{4}r^2 z\right]_{r=0}^{\sqrt{4-z}}$$

$$= \tfrac{2}{3}(4-z)^{3/2} + \tfrac{\pi}{4}(4-z)z - 0 = \tfrac{2}{3}(4-z)^{3/2} + \tfrac{\pi}{4}(4-z)z$$

Step 3: $$\int_{z=0}^{4} \left(\tfrac{2}{3}(4-z)^{3/2} + \tfrac{\pi}{4}(4-z)\underset{u}{z}\right)dz$$

$\left(\begin{matrix} u = 4-z \\ du = -dz \end{matrix}\right)$ $u(4)=0$, $u(0)=4$, $z = 4-u$

$$= -\int_{u=4}^{0} \left(\tfrac{2}{3}u^{3/2} + \tfrac{\pi}{4}u(4-u)\right)du$$

$$= \int_{u=0}^{4} \left(\tfrac{2}{3}u^{3/2} + \pi u - \tfrac{\pi}{4}u^2\right)du = \left[\tfrac{2}{3}\cdot\tfrac{2}{5}u^{5/2} + \tfrac{\pi}{2}u^2 - \tfrac{\pi}{12}u^3\right]_{u=0}^{4}$$

$$= \tfrac{4}{15}2^5 + \tfrac{\pi}{2}\cdot 16 - \tfrac{\pi}{12}\cdot 64 - 0 = \tfrac{128}{15} + \tfrac{3\cdot 8\pi}{3} - \tfrac{16\pi}{3}$$

$$= \tfrac{128}{15} + \tfrac{8\pi}{3} = \frac{128+40\pi}{15}$$

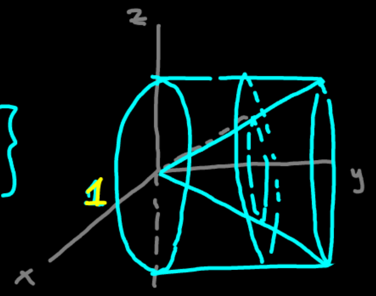
$$\therefore \iiint_R (x+y+z)\,dV = \frac{128+40\pi}{15}$$

$\square$

Ex: Compute $\iiint_R x^2\,dV$ for R the solid in cylinder $x^2+z^2\le 1$ to the right of the xz-plane and left of the cone $y^2=4x^2+4z^2$.

Sol: Use cylindrical coords $\begin{cases} x = r\cos(\theta) \\ y = y \\ z = r\sin(\theta) \end{cases}$ $\quad dV = r\,dV_{r\theta y}$

PICTURE



$y^2=4(x^2+z^2)$

$y^2=4r^2$

$\therefore\ y=2r$

$$R_{r\theta y}=\{(r,\theta,z):\ 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 0\le y\le 2r\}$$

$$\iiint_R x^2\,dV=\int_{r=0}^{1}\int_{y=0}^{2r}\int_{\theta=0}^{2\pi} r^3\cos^2(\theta)\,d\theta\,dy\,dr$$

Step 1: $\displaystyle\int_{\theta=0}^{2\pi} r^3\cos^2(\theta)\,d\theta$

$$=\frac{1}{2}r^3\int_{\theta=0}^{2\pi}\left(1+\cos(2\theta)\right)d\theta$$

$$=\frac{1}{2}r^3\left[\theta+\frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi}$$

$$=\frac{1}{2}r^3\left(\left(2\pi+\frac{1}{2}\cdot 0\right)-\left(0+\frac{1}{2}\cdot 0\right)\right)$$

$$=\pi r^3$$

Step 2: $\displaystyle\int_{y=0}^{2r}\pi r^3\,dy=\pi r^3\left[y\right]_{y=0}^{2r}$

$$=\pi r^3(2r-0)=2\pi r^4$$

Step 3: $\displaystyle\int_{r=0}^{1} 2\pi r^4\,dr=\frac{2}{5}\pi\left[r^5\right]_{r=0}^{1}$

$$=\frac{2}{5}\pi(1-0)=\frac{2}{5}\pi$$

Hence $\displaystyle\iiint_R x^2\,dV=\frac{2}{5}\pi$ ◻

Ex: Compute $\iiint_R xy\, dV$ for the region R given by

$$R = \left\{(x,y,z) : -2 \le z \le 2,\ -\sqrt{4-z^2} \le y \le \sqrt{4-z^2},\ \sqrt{y^2+z^2} \le x \le 2\right\}$$

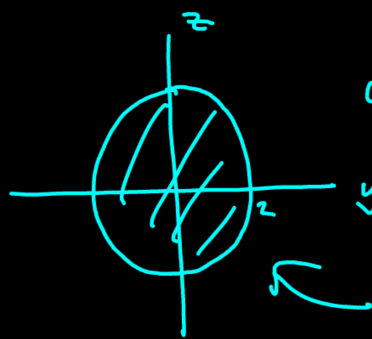
Sol: Use cylindrical coordinates $\begin{cases} x = x \\ y = r\cos(\theta) \\ z = r\sin(\theta) \end{cases}$ $\quad dV = r\, dV_{r\theta x}$

$y^2 + z^2 = r^2$

Parametrizing R: $r \le x \le 2$

$|y| \le \sqrt{4-z^2}$

$y^2 + z^2 \le 4$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$R = \{(r, \theta, x) : 0 \le r \le 2,\ r \le x \le 2,\ 0 \le \theta \le 2\pi\}$$

$$\therefore \iiint_R xy\, dV = \int_{r=0}^{2} \int_{x=r}^{2} \int_{\theta=0}^{2\pi} r^2 x \cos(\theta)\, d\theta\, dx\, dr$$

Step 1: $\int_{\theta=0}^{2\pi} r^2 x \cos(\theta)\, d\theta$

$= r^2 x \left[\sin(\theta)\right]_{\theta=0}^{2\pi}$

$= r^2 x \cdot 0 = 0$

Step 2: $\int_{x=r}^{2} 0\, dx = [0]_{x=r}^{2} = 0$

Step 3: $\int_{r=0}^{2} 0\, dr = [0]_{r=0}^{2} = 0$

$$\therefore \iiint_R xy\, dV = 0$$

$\blacksquare$

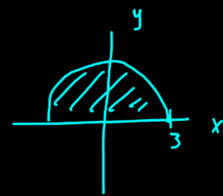
Ex: Compute $\iiint_R \sqrt{x^2+y^2}\, dV$ for R the solid region

$$R = \{(x,y,z) : -3 \le x \le 3,\ 0 \le y \le \sqrt{9-x^2},\ 0 \le z \le 9 - x^2 - y^2\}$$

Sol: Use cylindrical coordinates $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$ $\quad dV = r\, dV_{r\theta z}$

$$R = \{(x,y,z) : 0 \le y,\ x^2+y^2 \le 9,\ 0 \le z \le 9 - (x^2+y^2)\}$$

$$R_{r,\theta,z} = \{(r,\theta,z) : 0 \le \theta \le \pi,\ 0 \le r \le 3,\ 0 \le z \le 9 - r^2\}$$



$$\therefore \iiint_R \sqrt{x^2+y^2}\, dV = \int_{r=0}^{3} \int_{z=0}^{9-r^2} \int_{\theta=0}^{\pi} r\, r\, d\theta\, dz\, dr$$

Step 1: $\displaystyle\int_{\theta=0}^{\pi} r^2\, d\theta = r^2 [\theta]_{\theta=0}^{\pi} = \pi r^2$

Step 2: $\displaystyle\int_{z=0}^{9-r^2} \pi r^2\, dz = \pi r^2 [z]_{z=0}^{9-r^2} = \pi r^2 (9 - r^2)$

Step 3: $\displaystyle\int_{r=0}^{3} \pi (9r^2 - r^4)\, dr$

$$= \pi \left[3r^3 - \frac{1}{5} r^5\right]_{r=0}^{3}$$

$$= \pi \left(3 \cdot 3^3 - \frac{3^5}{5} - 0\right)$$

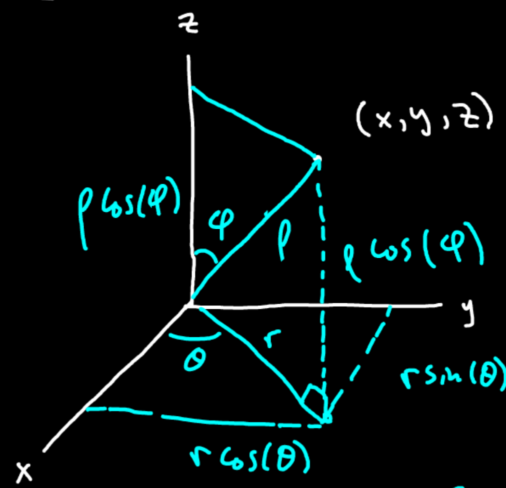
$$= \pi \left(81 - \frac{243}{5}\right) = \frac{405 - 243}{5} \cdot \pi = \frac{162}{5}\pi$$

$$\therefore \iiint_R \sqrt{x^2+y^2}\, dV = \frac{162}{5}\pi$$

$\square$

Spherical Coordinates

Idea: We build a transformation which parametrizes $\mathbb{R}^3$ in a natural way. In this new coordinate system, spheres about the origin are parametrized with constant bounds (cf. Circles in polar coordinates).



$r^2 + \rho^2\cos^2(\varphi) = \rho^2 \Rightarrow r = \rho\sin(\varphi)$

Thus the spherical coordinate transformation we calculated above is:

$$\begin{cases} x = \rho\sin(\varphi)\cos(\theta) \\ y = \rho\sin(\varphi)\sin(\theta) \\ z = \rho\cos(\varphi) \end{cases} \quad \text{where} \quad \begin{cases} \rho = \text{distance to the origin} \\ \theta = \text{angle of projected point w/ } \vec{i} \\ \varphi = \text{angle of point w/ } \vec{k} \end{cases}$$

To prepare for integration, we compute the Jacobian of this transform:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det\begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi} \end{bmatrix}$$

$$= \det\begin{bmatrix} \sin(\varphi)\cos(\theta) & -\rho\sin(\varphi)\sin(\theta) & \rho\cos(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta) & \rho\cos(\varphi)\sin(\theta) \\ \cos(\varphi) & 0 & -\rho\sin(\varphi) \end{bmatrix}$$

$$= \sin(\varphi)\cos(\theta)\begin{vmatrix} \rho\sin(\varphi)\cos(\theta) & \rho\cos(\varphi)\sin(\theta) \\ 0 & -\rho\sin(\varphi) \end{vmatrix}$$

$$+ \rho\sin(\varphi)\sin(\theta)\begin{vmatrix} \sin(\varphi)\sin(\theta) & \rho\cos(\varphi)\sin(\theta) \\ \cos(\varphi) & -\rho\sin(\varphi) \end{vmatrix}$$

$$+ \rho\cos(\varphi)\cos(\theta)\begin{vmatrix} \sin(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta) \\ \cos(\varphi) & 0 \end{vmatrix}$$

$$= \sin(\varphi)\cos(\theta)\left(-\rho^2\sin^2(\varphi)\cos(\theta)\right)$$

$$+ \rho\sin(\varphi)\sin(\theta)\left(-\rho\sin^2(\varphi)\sin(\theta) - \rho\cos^2(\varphi)\sin(\theta)\right)$$

$$+ \rho\cos(\varphi)\cos(\theta)\left(-\rho\sin(\varphi)\cos(\varphi)\cos(\theta)\right)$$

$$= -\rho^2\left(\sin^3(\varphi)\cos^2(\theta) + \sin^3(\varphi)\sin^2(\theta) + \sin(\varphi)\cos^2(\varphi)\sin^2(\theta) + \sin(\varphi)\cos^2(\varphi)\cos^2(\theta)\right)$$

$$= -\rho^2\sin(\varphi)\left(\sin^2(\varphi)\cos^2(\theta) + \sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)\cos^2(\theta)\right)$$

$$= -\rho^2\sin(\varphi)\left(\cos^2(\theta)\left(\sin^2(\varphi)+\cos^2(\varphi)\right) + \sin^2(\theta)\left(\sin^2(\varphi)\cos^2(\varphi)\right)\right)$$

$$= -\rho^2\sin(\varphi)\left(\cos^2(\theta) + \sin^2(\theta)\right) = -\rho^2\sin(\varphi)$$

Hence $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = -\rho^2\sin(\varphi)$.

NB: In spherical coordinates,

$0 \le \rho$, $0 \le \theta \le 2\pi$, and $0 \le \varphi \le \pi$

Hence $dA_{xyz} = \rho^2\sin(\varphi)\, dA_{\rho\theta\varphi}$ ✱

Ex: Compute $\iiint_R x^2 z \, dV$ for the region R given by

$$R = \left\{ (x,y,z) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2} \right\}$$

Sol: Use spherical Coordinates

$$\begin{cases} x = \rho \sin(\varphi)\cos(\theta) \\ y = \rho \sin(\varphi)\sin(\theta) \\ z = \rho\cos(\varphi) \end{cases}$$

$$dV_{xyz} = \rho^2 \sin(\varphi)\, dV_{\rho\varphi\theta}$$

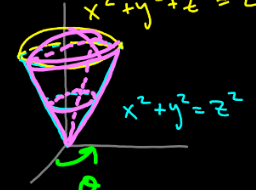
Bounds

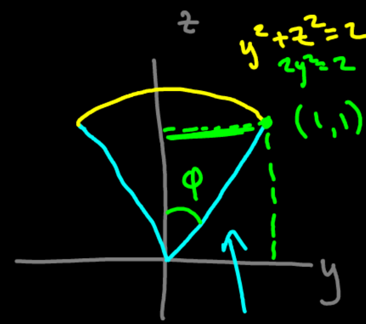
$$0 \le \rho \le \sqrt{2}$$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le \varphi \le \frac{\pi}{4}$$

PICTURE





$$\therefore R = \left\{ (\rho,\varphi,\theta) : \begin{matrix} 0 \le \rho \le \sqrt{2} \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{matrix} \right\}$$

$$\tan(\varphi) = \frac{y}{z} = 1 \qquad y^2 = z^2$$

$$\therefore \varphi = \frac{\pi}{4} \qquad \therefore y = z$$

$$\iiint_R x^2 z \, dV = \iiint_R \rho^2 \sin(\varphi)\ \rho^2\sin^2(\varphi)\cos^2(\theta)\ \rho\cos(\varphi)\ dV_{\rho\varphi\theta}$$

$$= \int_{\rho=0}^{\sqrt{2}} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\varphi=0}^{\pi/4} \rho^5 \cos^2(\theta)\sin^3(\varphi)\cos(\varphi)\, d\varphi\, d\theta\, d\rho$$

$$= \int_{\rho=0}^{\sqrt{2}} \rho^5 \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) \int_{\rho=0}^{\pi/4} \sin^3(\varphi)\cos(\varphi)\, d\varphi\, d\theta\, d\rho$$

Step 1: $\displaystyle\int_{\rho=0}^{\pi/4} \sin^3(\varphi)\cos(\varphi)\, d\varphi$

$$\begin{cases} u = \sin(\varphi) \\ du = \cos(\varphi)\, d\varphi \end{cases} \qquad u(\pi/4) = \frac{1}{\sqrt{2}},\quad u(0) = 0$$

$$= \int_{u=0}^{\frac{1}{\sqrt{2}}} u^3\, du = \left[\frac{1}{4}u^4\right]_{u=0}^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{4}\left(\left(\frac{1}{\sqrt{2}}\right)^4 - 0\right) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$$

Step 2: $\displaystyle\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{16}\cos^2(\theta)\, d\theta$

$$= \frac{1}{32}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1+\cos(2\theta)\right) d\theta$$

$$= \frac{1}{32}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{32}\left(\left(\frac{\pi}{2}+0\right)-\left(-\frac{\pi}{2}+0\right)\right)$$

$$= \frac{\pi}{32}$$

Step 3: $\displaystyle\int_{\rho=0}^{\sqrt{2}} \rho^5 \cdot \frac{\pi}{32}\, d\rho$

$$= \frac{\pi}{32}\left[\frac{1}{6}\rho^6\right]_{\rho=0}^{\sqrt{2}}$$

$$= \frac{\pi}{32}\cdot\frac{1}{6}\left(\sqrt{2}^6 - 0\right)$$

$$= \frac{\pi}{32}\cdot\frac{1}{6}\cdot 8$$

$$= \frac{\pi}{4}\cdot\frac{1}{6}$$

$$= \frac{\pi}{24}$$

Hence $\displaystyle\iiint_R x^2 z\, dV = \frac{\pi}{12}$ ☒

Ex: Compute $\int_{y=-\alpha}^{\alpha}\int_{x=-\sqrt{\alpha^2-y^2}}^{\sqrt{\alpha^2-y^2}}\int_{z=-\sqrt{\alpha^2-x^2-y^2}}^{\sqrt{\alpha^2-x^2-y^2}}(x^2z+y^2z+z^3)\,dz\,dx\,dy$.
where $\alpha>0$.

Sol: Note $R=\left\{(x,y,z):\begin{array}{l}-\alpha\le y\le\alpha,\ -\sqrt{\alpha^2-y^2}\le x\le\sqrt{\alpha^2-y^2},\\ -\sqrt{\alpha^2-x^2-y^2}\le z\le\sqrt{\alpha^2-x^2-y^2}\end{array}\right\}$

is a sphere of radius α about the origin.

$\therefore\ R=\{(\rho,\varphi,\theta):0\le\rho\le\alpha,\ 0\le\varphi\le\pi,\ 0\le\theta\le 2\pi\}$ ← spherical coordinates!

$$\left\{\begin{array}{l}x=\rho\sin(\varphi)\cos(\theta)\\ y=\rho\sin(\varphi)\sin(\theta)\\ z=\rho\cos(\varphi)\end{array}\right. > \quad dV_{xyz}=\rho^2\sin(\varphi)\,dV_{\rho\varphi\theta}$$

Note $x^2z+y^2z+z^3=(x^2+y^2+z^2)z=\rho^2\cdot\rho\cos(\varphi)=\rho^3\cos(\varphi)$

$$\therefore\iiint_R z(x^2+y^2+z^2)\,dV=\iiint_R \rho^2\sin(\varphi)\rho^3\cos(\varphi)\,dV_{\rho\varphi\theta}$$

$$=\int_{\rho=0}^{\alpha}\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi}\rho^5\cos(\varphi)\sin(\varphi)\,d\theta\,d\varphi\,d\rho$$

$$=\int_{\rho=0}^{\alpha}\int_{\varphi=0}^{\pi}\rho^5\cos(\varphi)\sin(\varphi)\,\Big[\theta\Big]_{\theta=0}^{2\pi}\,d\varphi\,d\rho$$

$$=2\pi\int_{\rho=0}^{\alpha}\int_{\varphi=0}^{\pi}\rho^5\cos(\varphi)\sin(\varphi)\,d\varphi\,d\rho$$

$$=2\pi\int_{\rho=0}^{\alpha}\int_{u=0}^{0}\rho^5\,u\,du\,d\rho \qquad \left(\begin{array}{ll}u=\sin(\varphi) & u(\pi)=0\\ du=\cos(\varphi)d\varphi & u(0)=0\end{array}\right.$$

$$=2\pi\cdot 0=0$$

$\blacksquare$

Ex: Compute the volume of the general ellipsoid with formula

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1 \quad \text{where } a,b,c>0.$$

Sol: Make a coordinate transformation informed by spherical coordinates...

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

$$\Leftrightarrow \left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 + \left(\frac{z-z_0}{c}\right)^2 = 1 \quad \text{suggests}$$

$$\begin{cases} \frac{x-x_0}{a} = \rho\sin(\varphi)\cos(\theta) \\ \frac{y-y_0}{b} = \rho\sin(\varphi)\sin(\theta) \\ \frac{z-z_0}{c} = \rho\cos(\varphi) \end{cases} \Leftrightarrow \begin{cases} x = a\rho\sin(\varphi)\cos(\theta) + x_0 \\ y = b\rho\sin(\varphi)\sin(\theta) + y_0 \\ z = c\rho\cos(\varphi) + z_0 \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \det\begin{bmatrix} x_\rho & x_\varphi & x_\theta \\ y_\rho & y_\varphi & y_\theta \\ z_\rho & z_\varphi & z_\theta \end{bmatrix}$$

$$= \det\begin{bmatrix} a\sin(\varphi)\cos(\theta) & a\rho\cos(\varphi)\cos(\theta) & -a\rho\sin(\varphi)\sin(\theta) \\ b\sin(\varphi)\sin(\theta) & b\rho\cos(\varphi)\sin(\theta) & b\rho\sin(\varphi)\cos(\theta) \\ c\cos(\varphi) & -c\rho\sin(\varphi) & 0 \end{bmatrix}$$

← EEK!

$$= a\rho\sin(\varphi)\cos(\theta)\big(0 + b\rho\sin(\varphi)\cos(\theta)\cdot + c\rho\sin(\varphi)\big)$$

$+ a\rho\cos(\varphi)\cos(\theta)\left(0 + b\rho\sin(\varphi)\cos(\theta)\cdot c\cos(\varphi)\right)$

$+ + a\rho\sin(\varphi)\sin(\theta)\left(b\sin(\varphi)\sin(\theta)\cdot c\rho\sin(\varphi) + b\rho\cos(\varphi)\sin(\theta)\cdot c\cos(\varphi)\right)$

$= abc\rho^2\sin(\varphi)\left(\cos^2(\theta)\left(\sin^2(\varphi)+\cos^2(\varphi)\right) + \sin^2(\theta)\left(\sin^2(\varphi)+\cos^2(\varphi)\right)\right)$

$= abc\rho^2\sin(\varphi)\left(\cos^2(\theta)+\sin^2(\theta)\right)\left(\sin^2(\varphi)+\cos^2(\varphi)\right)$

$= abc\rho^2\sin(\varphi)$

NB: $R = \left\{(\rho,\varphi,\theta) : \begin{array}{l} 0\le\rho\le 1 \\ 0\le\theta\le 2\pi \\ 0\le\varphi\le\pi \end{array}\right\}$

$\therefore V = \iiint_R 1\, dV$

$= \iiint_R abc\rho^2\sin(\varphi)\, dV_{\rho,\theta,\varphi}$

$= abc\int_{\rho=0}^{1}\rho^2\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\sin(\varphi)\, d\varphi\, d\theta\, d\rho$

Step 1: $\int_{\varphi=0}^{\pi}\sin(\varphi)\, d\varphi = -\left[\cos(\varphi)\right]_{\varphi=0}^{\pi} = -(-1-1) = 2$

Step 2: $\int_{\theta=0}^{2\pi} 2\, d\theta = 2\left[\theta\right]_{\theta=0}^{2\pi} = 2(2\pi - 0) = 4\pi$

Step 3: $\int_{\rho=0}^{1} 4\pi\rho^2\, d\rho = \frac{4}{3}\pi\left[\rho^3\right]_{\rho=0}^{1} = \frac{4}{3}\pi(1-0) = \frac{4}{3}\pi$

Hence $\text{Vol} = abc\cdot\frac{4}{3}\pi = \frac{4}{3}\pi abc$ $\square$

Cor: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

pf: The sphere of radius r is an ellipsoid with equation $\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=1$, so $a=b=c=r$ in the formula :)